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ON THE USE OF COORDINATE STRETCHING IN THE NUMERICAL COMPUTATION OF HIGH FREQUENCY SCATTERING

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ABSTRACT

The scattering of the sound of a jet engine by an airplane fuselage is modelled by solving the axially symmetric Helmholtz equation exterior to a long thin ellipsoid. The integral equation method based on the single layer potential formulation is used. A family of coordinate systems on the body is introduced and an algorithm is presented to determine the optimal coordinate system. Numerical results verify that the optimal choice enables the solution to be computed with a grid that is coarse relative to the wavelength.

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I. Introduction

The problem considered here is the numerical computation of the far field scattering of scalar waves generated by a source at a point q on the axis of symmetry of an elongated ellipsoid, E (see figure 1). The physical problem that motivated this work is the scattering of the sound of a jet engine by an airplane fuselage which is modelled by the ellipsoid.

If ∂E denotes the boundary of E and p represents the independent variables, one is interested in the solution to the following exterior Neumann problem:

$$(1.1)\quad \text{a.}\quad \Delta v + k^2 v = -\delta(p-q)$$

$$\text{b.}\quad v_n = 0 \text{ on } \partial E$$

$$\text{c.}\quad \lim_{R\to\infty} \int |v_r - ikv|^2 \, dA \to 0 \qquad (R\to\infty)$$

where the integral in (1.1c) is over spherical shells of radius R and k is the reciprocal wavelength ($2\pi f/c$ where f is the frequency and c the sound speed).

It is well known (see Courant-Hilbert, Volume 2, [1]) that the imposition of the radiation condition (1.1c) makes the problem (1.1) well posed. In order to solve (1.1) numerically one writes the solution v as $u^1 + u$ where u is smooth in the exterior of E, which we call E', and u^1 is the singular part

$$(1.2)\qquad u^1(p) = \frac{1}{4\pi}\,\frac{e^{ik|p-q|}}{|p-q|}\,.$$

We then obtain the exterior Neumann problem for u which we call problem A:

(1.3) a. $\Delta u + k^2 u = 0$

b. $u_n = -u_n^1$ on ∂E

c. $\| u_r - iku \|_R \to 0 \quad (R \to \infty)$

This paper is concerned with obtaining far field solutions to the problem (1.3). This problem can be solved explicitly as an infinite series (see Bowman [2]) but the series converges slowly unless k is small, and is not suited to numerical computation for intermediate frequencies. Asymptotic expansions are available for large values of k (see [2] and Levy and Keller [3]) but these will generally not be uniformly valid for all regions in the far field. There is a large frequency range where one can not obtain accurate solutions for certain regions of the far field and for these frequencies direct numerical methods must be applied.

The problem is first formulated as an integral equation by using a technique discussed by various authors (see Kellogg [4], Copley [5] and Burton [6]). One assumes the solution has the form

$$u(p) = \int_{\partial E} \sigma(q)\, G(p,q)\, dA_q \tag{1.4}$$

where

$$G(p,q) = \frac{1}{4\pi} \frac{e^{ik|p-q|}}{|p-q|}$$

is the free space Green's function and where the unknown potential σ is defined on the surface ∂E. On taking the derivative of (1.4) along the outward normal to the surface ∂E and letting the point p approach ∂E one obtains (see [6])the surface Fredholm equation of the second kind

$$\frac{\sigma(q)}{2} - \int dA_{q'} \sigma(q') G_{n_q}(q,q') = -u_n(q) \tag{1.5}$$

We consider only the case of axially symmetric Neumann data so that (1.5) reduces to a one-dimensional equation.

The formulation expressed by (1.4) and (1.5) has several problems associated with it. It is known that this formulation will break down at the eigenvalues of the interior Dirichlet problem (see [6]). In general, at these frequencies the equation (1.5) does not have a solution and discretizations for nearby frequencies will be numerically ill-conditioned. An extensive discussion of techniques of overcoming this problem is given in [6]. It is the experience of the author that the ill-conditioning due to the interior resonances does not extend over a wide frequency range and this problem will not be considered further.

A more fundamental problem is that of adequately resolving the solution σ at high frequencies. It can be seen from the form of the fundamental solution $G(p,q)$ that the Neumann data becomes very oscillatory at high frequencies and it is found that the potential itself is also highly oscillatory. The resolution problem requires that one obtain a grid which not only adequately resolves the solution σ but also resolves the Green's function $G(p,q)$ so that the quadrature (1.4) can be accurately computed. It will be seen that this is much more crucial for field points lying near the major axis than near the minor axis as the Green's function is much less oscillatory in the latter case.

In this paper we introduce a family of new angular coordinates on the ellipsoid depending on a free parameter α. We denote this family by ϕ_α. These coordinates are related to the polar coordinate angle θ by

$$\theta = \tan^{-1}(\alpha \tan \phi_\alpha) \tag{1.6}$$

When α is the ratio of major to minor axes the new coordinates are the elliptical angular coordinates (see [2]). It will be seen that these

coordinates are generally unsatisfactory when the grid is coarse relative to the wave length. If, however, α is properly chosen, accurate high frequency solutions can be obtained with a grid that is coarse relative to the wave length and which would give unsatisfactory results with an evenly spaced grid in θ.

A discussion of the method of choosing α is given in section 3. It will be shown in the appendix that if the aspect ratio e is large the choice

$$\alpha = e\,\frac{\sqrt{2}}{2} \tag{1.7}$$

is nearly optimal. It is further shown that (1.7) is independent of the source position.

The study of the resolution problem is made difficult because no exact solution is effectively available to judge the accuracy of different schemes. If the source point q is near the tip, however, a solution with similar oscillatory properties can be obtained. In fact if we let q' be the point symmetrical to q but interior to E, we will denote by Problem B the following problem

$$\text{a.}\quad \Delta u + k^2 u = 0 \qquad p\in E'$$

$$\text{b.}\quad u_n(p) = -\frac{\partial}{\partial n_p}\,\frac{e^{ik|p-q'|}}{|p-q'|} \qquad p\in\partial E \tag{1.8}$$

$$\text{c.}\quad \| u_r - iku \|_R \to 0 \quad (R\to\infty)$$

Problem B has the exact solution

$$u(p) = \frac{e^{ik|p-q'|}}{|p-q'|} \qquad p\in E'$$

as can easily be verified. Examination of the Neumann data for the two problems shows that the oscillatory properties are similar. The

availability of this exact solution permits a detailed evaluation of the accuracy of various schemes and parameter choices.

In section 2 we give the details of the numerical scheme used used to solve (1.5). In section 3 we describe how to choose the proper stretching parameter α. Numerical results are presented and a discussion of the oscillatory properties of the Green's function in (1.4) is given. In section 4 we compare numerical computations for the scattering problem with the asymptotic expansion of geometrical optics. It is found that the accuracy of this expansion depends on the size of the scattered field and the angle of observation. The convergence of the two solutions as ka increases will be apparent.

II. Details of the Numerical Scheme

We first establish our notation. We denote by a and b the semi-major and semi-minor axes of E. The aspect ratio is defined as the ratio a/b. The quantity k has units $(\text{length})^{-1}$ and in discussing frequencies we will use the non-dimensional quantity ka. If the coordinates are scaled by a constant factor a the solution to (1.1), and k are similarly scaled and thus the actual value of a is irrelevant.

We will use θ to denote the polar angle and we will define $r = g(\theta)$ as the polar coordinate representation of the ellipse which is rotated to give the ellipsoid. The azimuthal angle will be denoted by λ. New angular coordinates will be denoted by ϕ_α where the parameter α is given in (1.6), or just by ϕ if no confusion will arise. Far field solutions will be computed on a large circle surrounding E. By θ_0 we will denote the polar angle of the observation point.

The equation (1.5) is reduced to a 1-dimensional equation in the polar angle θ. Discretization of the integral in (1.5) at n points gives rise to an nxn linear system. The discretization is done by the trapezoidal rule using an evenly spaced grid in the new coordinates ϕ (see (1.6)).

Higher order formulas were found to be unsatisfactory when the grid was coarse relative to the wavelength $1/k$. This is because higher order formulas, with a grid spacing h, will have errors of the from $(hk)^{p+1}$ for some p (see Isaacson and Keller [7]) and hk is not small.

The kernel in (1.5) has an integrable singularity at $q = q'$. The behavior near the singularity is the same as for Laplace's equation ($k = 0$) and is treated by a technique described in [6]. One uses the fact that

$$\frac{\partial}{\partial n_q} G(q,q') = \frac{\partial}{\partial n_{q'}} G^0(q,q') + o(1) \qquad (q \to q')$$

where

$$G^0(q,q') = (4\pi|q-q'|)^{-1}$$

is the free space Green's function for the Laplace equation. From Green's theorem one has

$$\int_{\partial E} dAq' \frac{\partial}{\partial n_{q'}} G^0(q,q') = -\tfrac{1}{2}$$

and this enables one to compute the integral of the kernel across the singularity.

Using the axial symmetry, the equation (1.5) is rewritten as a 1-dimensional equation

$$\frac{\sigma(\theta)}{2} - \int_{-\pi/2}^{\pi/2} d\bar{\theta}\ \sigma(\bar{\theta}) H(\theta,\bar{\theta}) = -u_n(\theta)\ . \tag{2.1}$$

The kernel function is given explicitly by

$$H(\theta,\bar{\theta}) = \cos\bar{\theta}\ g(\bar{\theta})\sqrt{g(\bar{\theta})^2+g'(\bar{\theta})^2} \int_0^{2\pi} d\lambda \frac{\partial}{\partial n_{\bar{q}}} G(q,\bar{q})$$

where the point $\bar{q}$ corresponds to the spherical coordinates $(\bar{\theta},\lambda)$, and the point q corresponds to the spherical coordinates $(\theta,0)$. Introduction of new coordinates ϕ adds only the term $\frac{d\theta}{d\phi}$ multiplying H. The numerical integration in the λ direction is done using a grid so that $\Delta\lambda \cos\theta$ is constant. This integration is not part of the discretization and does not affect the size of the matrix.

If p denotes a far field point, the solution $u(p)$ is computed by numerical integration of (1.4). If we denote by $\tilde{G}(p,\theta)$ the longitudinally integrated Green's Function.

$$\tilde{G}(p,\theta) = \int_0^{2\pi} d\lambda \, \frac{e^{ik|p-q|}}{4\pi|p-q|} \tag{2.3}$$

where q is the point on the ellipsoid with spherical coordinates (θ,λ) then

$$u(p) = \int_{-\pi/2}^{\pi/2} d\theta \; \sigma(\theta)\tilde{G}(p,\theta)g(\theta)\sqrt{g^2(\theta)+g'^2(\theta)} \tag{2.4}$$

The same grid is used as in the solution of (2.1) introducing the factor $\frac{d\theta}{d\phi}$ into (2.4). We point out that the transformation (1.6) preserves the symmetry about the line $\theta = 0$ and this reduces by one-half the work involved in computing the matrix corresponding to the discretization of (1.5). This is by far the costliest part of the computation. Typical running times on the Cyber 175, for a grid of 129 points are 40 seconds to compute the matrix and 1.8 seconds for the Gaussian elimination. The computation of the far field from the numerical integration of (1.4) requires .5 seconds per point and thus the method becomes more efficient the greater the number of points required.

III. Choice of Stretching Parameter

In order to solve the system (1.4 - 1.5) with a coarse grid, relative to k, one must make an appropriate choice of the new coordinates ϕ, or equivalently the parameter α (see (1.6)).

It is possible to constructively determine the optimal parameter. To do this we refer to the integral equation (2.1). Let ϕ be any angular coordinate system on the ellipse and if q is the source point for either problem A or problem B we let $R_q(\phi)$ be the distance from the axial point q to the point corresponding to the angle ϕ on the ellipse. The right hand side of (2.1) will be of the form

$$u_n(\phi) = [\text{smooth function of } \phi] \times e^{ikR_q(\phi)} \tag{3.1}$$

and thus the oscillatory behavior will be that of the function $e^{ikR_q(\phi)}$. If ϕ_α belongs to a family of coordinate systems depending on α it is clearly necessary to choose the system for which $e^{ikR_q(\phi)}$ is the smoothest.

Locally, about any point ϕ^0, one has

$$R_q(\phi) \cong R_q(\phi^0) + R_q'(\phi_0)\,(\phi-\phi^0)$$

and thus

$$e^{ikR_q(\phi)} \sim e^{ikR_q(\phi^0)}\; e^{ikR_q'(\phi^0)\,(\phi-\phi^0)} \tag{3.2}$$

Equation (3.2) can be interpreted as giving rise to a local wave number

$$n(\phi^0) = kR_q'(\phi^0) \tag{3.3}$$

and the point ϕ^0 where $|R_q'|$ is maximized determines the point of largest local wave number, or equivalently the point where (3.1) is most oscillatory. It is therefore clear that in order to obtain maximal resolution of (3.1)

with a fixed evenly spaced ϕ grid, one should choose α so that the maximum of $|R_q'(\phi)|$ is minimized, i.e., α satisfies a mini-max principal

$$\min_{\alpha'} \max_{\phi} |R_q'(\phi_{\alpha'})| \tag{3.4}$$

For an elongated ellipsoid with aspect ratio $e >> 1$ the mini-max problem (3.4) has a solution

$$\alpha = e\left(\frac{\sqrt{2}}{2} + 0(e^{-2})\right) \tag{3.5}$$

where the remainder in (3.5) is independent of the source position. The exact solution of (3.4) is generally not required and the choice

$$\alpha = e\,\frac{\sqrt{2}}{2}$$

is sufficient in the tests which will be described below.

It is found that α determined by (3.4) always provides optimal or nearly optimal solutions. To demonstrate this we present numerical results for the test problem B for two different ellipsoids. Ellipsoid 1 has an aspect ratio of 7.083 and is of direct interest as a model of an airplane fuselage. Ellipsoid 2 has an aspect ratio of 3 and is used to show that the stretching transformation (2.2) is not restricted to thin ellipsoids.

The results tabulated below are for a non-dimensionalized frequency ka of 150. In both cases a grid of 129 points was used. The interior "source" point q' was along the major axis at a distance of .96a and the field is computed on a sphere of radius of 20.2a.

In table 1 we present L_2 errors for ellipsoid 1 for different values of α. The maximum of $|R_q'(\phi)|$ is also indicated.

α	max $\lvert R_q' \rvert$	L_2 error
4.7	16.3	.013
5.0	16.2	.011
5.5	16.54	.012
6.0	17.8	.178
6.8	20.2	3.51
7.0833	21.0	2.14
7.5	22.3	5.72

TABLE 1.

In table 2 similar results are presented for ellipsoid 2.

α	max $\lvert R_q' \rvert$	L_2 error
2.0	15.8	.11
2.2	15.6	.092
2.4	16.1	.076
2.75	18.4	4.18
3.0	20.1	2.84
3.3	22.1	2.88

TABLE 2.

In both cases we note that stretching based on elliptical coordinates (i.e., α = aspect ratio) gives inaccurate results while if α is chosen to satisfy (3.4) accurate results can be obtained.

A more detailed examination of the solutions which generated tables 1 and 2 shows that in all cases the solutions agree closely near the minor axis and that large errors occur for observation direction near the major axes. This is due to the nature of the Green's function appearing in the quadrature (1.5).

It is found the longitudinally integrated Green's function (2.3) is much more oscillatory for field points p near the major axis than the minor axis. If p is along the major axis the Green's function is just

$$\frac{e^{ikR_p(\phi)}}{4\pi R_p(\phi)}$$

where R_p is the distance from the point p to the point described by ϕ on the ellipsoid. The results presented in the appendix show that the solution to (3.4) is independent of the fixed axial point p and thus the

coordinate system obtained will also provide adequate resolution for the quadrature (1.4). As the body approaches a sphere, however, this is not the case and this limits the accuracy which can be obtained for field points near the major axis.

The principles described here have been used to generate solutions for the scattering problem A. Comparing solutions for different grid sizes, it has been found that determining α by the principal (3.4) enables one to compute accurate solutions with grids that are much coarser than those required for the natural coordinate system of the elliptical angular variables.

IV. Comparisons With Geometrical Optics

For large values of ka asymptotic expansions for the solution to problem A are known. The most common expansion is that of geometrical optics (described concisely in [2], pp. 22-24 and discussed in greater generality in [3]). Here we do not attempt to describe the theory but rather to judge the validity of this expansion as compared to accurate solutions generated by the integral equation method.

Referring to figure 2, geometrical optics predicts the field at the point p as the superposition of the incident field and the scattered field reflected from the point s. It is apparent from the figure that there is a region where no reflected rays can reach. This is called the shadow region and the geometrical optics field has a discontinuity between the shadow region and its complement (the illuminated region). In the notation of figure 2 the geometrical optics field can be written as

$$v_{g0}(p) = \frac{e^{ik|q-p|}}{4\pi|q-p|} + z(p)\frac{e^{ik|s-p|}}{|s-p|} \qquad p \in \text{illuminated region} \tag{4.1a}$$

$$v_{g0}(p) = 0 \qquad p \in \text{shadow region} \tag{4.1b}$$

The amplitude function $z(p)$ is determined by a principle of "conservation of energy" along tubes of rays and is given explicity in [2] (p. 24). We point out that (4.1) must be regarded as an approximation to the full problem (1.1) and not for the solution u to (1.3). This will become apparent from the results presented below.

Computations were made for ellipsoids 1 and 2 for different source positions and different values of ka. Solutions were computed at 10° intervals on a circle of radius 20.2a and the integral equation solutions for the scattered field are believed accurate to 10%. L_2 differences were computed and normalized by both the total and scattered field.

Results are first presented for $\theta_0 \geq 0$ where geometrical optics is most accurate. Table 3 shows the L_2 differences for ellipsoid 1 for the source at 1.04a. The second entry, E_1, is the L_2 difference normalized by the total field while the third entry, E_2, has normalization by just the scattered field. The fourth entry, R_1, is the ratio of the L_2 norm of the scattered field to the L_2 norm of the total field.

ka	E_1	E_2	R_1
75	.122	.720	.16
100	.096	.576	.16
125	.088	.545	.18
150	.083	.414	.20
200	.060	.316	.19

TABLE 3.

We observe that the errors normalized by the total field are very small and do not reflect the large errors when normalized by just the scattered field. This is because geometrical optics is an approximation to the total field and will not accurately approximate the scattered field when it is only a small component of the total field.

This is further demonstrated in table 4 where the same quantities are presented for ellipsoid 2. Here the scattered field is a much greater component

of the total field and we see that the errors are relatively insensitive to the normalization

ka	E_1	E_2	R_1
125	.088	.185	.47
150	.082	.148	.48

TABLE 4.

Finally we consider the accuracy of the approximation (4.1) as the shadow region is approached. Table 5 shows the same quantities but with the region $\theta_0 \geq -60^0$ used for the comparison. All points are in the illuminated region but the degradation as one approaches the shadow region is apparent.

ka	E_1	E_2	R_1
75	.176	1.22	.14
100	.154	1.01	.15
125	.149	.903	.17
150	.147	.795	.18
200	.124	.675	.18

TABLE 5.

In table 6 the same quantities are given with the errors taken over the entire region. The very large inaccuracy of (4.1b) is clearly evident.

ka	E_1	E_2	R_1
75	.288	1.21	.24
100	.263	1.01	.26
125	.260	.853	.30
150	.253	.740	.34
200	.221	.632	.34

TABLE 6.

The conclusion drawn is that the validity of the geometrical optics approximation depends on the size of the scattered field relative to the total field and on whether the region of interest is bounded away from the shadow region. In the shadow region the theory of geometrical diffraction (see [2], Chapter 11) has been tested and found very inaccurate for elongated ellipsoids.

An experiment to verify these results has been conducted and is reported on separately (see Bayliss and Maestrello [8]). Close agreement was obtained using ellipsoid 1 for values of ka up to 166, the limit of the experimental apparatus. Figure 3 is a polar plot of the sound pressure level (relative to the axis) for $ka = 166$. Also plotted are the experimental values and the results predicted by geometrical optics. The divergence of the two methods as the shadow region is approached is evident.

It is felt that the principle of minimizing the local wave number as outlined in section 3 provides a method for extending the range of frequencies for which scattering can be computed numerically for a given amount of work. The method appears to be general, at least for bodies which do not have edges or cusps and although typically the mini-max principle (3.4) can only be solved numerically, this could make substantial improvements in the efficiency of the numerical calculations.

Appendix

Here we give a proof of relation (3.5) in the determination of the optimal parameter α.

Letting θ_e denote the elliptical angular coordinates we have

$$\theta = \tan^{-1}(e \tan \theta_e) \ .$$

Thus if γ is defined by

$$\theta_e = \tan^{-1}(\gamma \tan \phi) \tag{A-1}$$

it follows that

$$\alpha = e\gamma$$

and it is thus sufficient to show that

$$\gamma = \frac{\sqrt{2}}{2} + 0(e^{-2}) \ . \tag{A-2}$$

For simplicity we will set $a = 1$ so that $e = 1/b$ where b is the semi-minor axis. We can then rewrite equation (A-2) as

$$\gamma = \frac{\sqrt{2}}{2} + 0(b^2) \ . \tag{A-3}$$

If z_0 denotes the source position, $(z_0 > 1)$ we have

$$R_{z_0}(\theta_e) = \sqrt{(\sin \theta_e - z_0)^2 + b^2 \cos^2 \theta_e}$$

for the distance function from the source to the body. Dropping the subscript z_0 and differentiating we obtain

$$\frac{dR}{d\theta_e} = \frac{\cos \theta_e [\sin \theta_e (1-b^2) - z_0]}{R} \ ,$$

which can be written as

$$\frac{dR}{d\theta_e} = \cos\theta_e(1+0(b^2)) \tag{A-4}$$

with the reminder uniform in θ_e and uniform in z_0 for z_0 bounded away from 1.

From (A-1) we obtain

$$\phi = \tan^{-1}(\gamma^{-1}\tan\theta_e)$$

and on differentiating we have

$$\frac{d\theta_e}{d\phi} = \left(\frac{d\phi}{d\theta_e}\right)^{-1} = (\gamma^2\cos^2\theta_e + \sin^2\theta_e)/\gamma \quad . \tag{A-5}$$

Letting $w = \sin\theta_e$ the mini-max principle of (3.4) can now be stated as

$$\min_{0\le\gamma} 1/\gamma \max_{|w|\le 1} \left[\sqrt{1-w^2}\ (1+0(b^2))(\gamma^2(1-w^2)+w^2)\right]. \tag{A-6}$$

If we denote by $m(\gamma,b)$ the maximum of the term in brackets in (A-6) and by $T(\gamma,b)$ the quotient $m(\gamma,b)/\gamma$ it follows by straightforward differentiation that

$$m(\gamma,0) = \begin{cases} \dfrac{2}{3\sqrt{3}\ \sqrt{1-\gamma^2}} & , \quad \gamma^2 \le \dfrac{2}{3} \\ \gamma^2 & , \quad \gamma^2 \ge \dfrac{2}{3} \end{cases} \tag{A-7}$$

and thus it is easy to see that $T(\gamma,0)$ has a unique minima at $\gamma = \gamma^* = \frac{\sqrt{2}}{2}$ with $\dfrac{d^2T}{d\gamma^2}(\gamma^*,0) > 0$.

Now for small b we have for some constant C

$$(1-Cb^2)T(\gamma,0) \le T(\gamma,b) \le T(\gamma,0)[1+Cb^2]$$

and it follows from the fact that the $\frac{d^2T}{d\gamma^2}(\gamma^*,0)$ is non-zero that the minima of $T(\gamma,b)$, which we denote by γ_b satisfies

$$|\gamma_b - \gamma^*| = 0(b^2)$$

which was to be proved.

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